Nonlinear relaxation in the presence of an absorbing barrier

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We study the nonlinear relaxation in the presence of multiplicative noise by means of a simple approximation scheme valid outside the critical region and exact asymptotic expansion at the critical point. The theory is developed in the Malthus-Verhulst stochastic model case. We find nonmonotonic growth of fluctuations during the transient. At the critical point we study the statistical properties of the finite time average of the original process. We obtain an exact result for the generating function exhibiting scaling asymptotic behavior at the critical point. We deduce also an asymptotic sum rule for the n-times correlation function of the original process and the asymptotic expression of the two-times correlation function. Our theoretical results are compared with numerical simulations and steady-state known properties.

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I. INTRODUCTION

The effect of multiplicative noise has been investigated recently from both an experimental and a theoretical point of view. Relevant experiments have been performed on laser radiation fluctuations [1,2]. Stochastic models with state-dependent noise have been introduced to explain a wide class of physical process such as nonequilibrium transitions in liquid crystals [3], vacuum fluctuations in quantum optics [4], the statistics of multifractal objects [5], and the previously quoted light statistics in dye lasers. A general reference reporting both theoretical and experimental results may be found in Ref. [6]. Theoretical analysis of the steady-state statistical properties of the laser radiation in the presence of the multiplicative noise has been performed in connection with experiments in Ref. [7].

Nonmonotonic growth of fluctuations in a nonlinear relaxation from a definite initial state has been discussed in Ref. [8] in the case in which the initial state is far away from both the absorbing barrier and the equilibrium state. This phenomenon can be understood considering that the system relaxes from a definite initial state under the influence of noise which decreases in amplitude as long as the equilibrium state is approached.

The new feature of the transient behavior which is discussed in the present paper, is the occurrence of the anomalous fluctuations even in the case of an initial state close to the absorbing barrier. This phenomenon is similar to that observed in the decay of an unstable state under the action of an additive noise [9]. The appearance of the anomalous fluctuations in the decay from an initial state close to the absorbing barrier is due to the amplification of an initially small fluctuation during the transient. An important difference in the relaxation process in the presence of additive or multiplicative noise is the amplitude of the anomalous fluctuations which is smaller in the latter case than in the former. The usual approach to anomalous fluctuations in the presence of additive noise [9] is, however, ineffective in the present case. This is because the additive noise is relevant only during the early stages of relaxation, while the multiplicative noise is vanishingly small near the absorbing barrier and increases as the equilibrium state is approached.

A simple approximation scheme can be introduced in two steps. First a mapping of the original process into an additive noise process is introduced and then the new process is approximated by a Gaussian process. This approximation is valid far from the critical point. The mapping has been already introduced in the case of the decay from an initial state which is far away both from equilibrium and the absorbing state [8].

In the region of parameters where the decay occurs towards the absorbing barrier the Gaussian approximation does not work because the additive noise process (and its fluctuations) diverges as the original process approaches the absorbing barrier. In order to overcome this difficulty we found it convenient to study the finite time integral of the original process. This process attains a finite value once the original process reaches the absorbing barrier. Thus a small fluctuation regime for the time-average process is expected and the Gaussian approximation works.

All our considerations will be developed in a simple case: the Malthus-Verhulst stochastic model (MVSM) already considered in the literature [8,10,11].

It is worth noting that the solution of the associated stochastic differential equations is known in this case and exact expressions for the moments have been developed by several authors [10,12]. These expressions are, however, quite complicated and even difficult to evaluate by numerical methods [13]. An approach that improves convergence of a series expressing the moments of the exact analytic solution has been introduced in Ref. [11]. More recently the first-passage-time statistics has been studied in a more general model which includes the presence of additive noise using the small noise expansion [14]. The results of the latter work are strongly dependent of the noise strength in the region of the parameter space in which we expect the anomalous fluctuations to the relevant feature of the transient.

A separate analysis is devoted to the critical slowing down at the critical point. We show that the time integral process asymptotically exhibits a scaling behavior at the critical point. This asymptotic behavior is derived first rigorously and then from a heuristic argument. We want to emphasize that this heuristic procedure give us an alternative derivation of the noise-induced long-time tail of the moments of the original process derived by Suzuki and co-workers [10]. Moreover, the knowledge of the dynamical behavior of the moments of the time-averaged process can give us information about the properties of the n-times correlation functions of the original process.

The paper is organized as follows. In Sec. II we introduce the model and discuss the difficulties of the usual self-consistent approximation. This approximation is unable to control large (even if rare) fluctuations, which are characteristic of state-dependent noise.

In Sec. III the growth of anomalous fluctuations for an initial state close to the absorbing barrier is studied by means of linearization of the model equations with respect to deterministic evolution.

In Sec. IV the mapping is introduced. We see that a simple linearization around the deterministic evolution of the additive noise process provides an accurate approximation outside the critical region. Fluctuation behavior in the case of the decay from an initial state far away from equilibrium and the absorbing states is easily obtained in the framework of this approximation.

The additive noise process does not help, as we discuss in Sec. V, in the case of decay towards the absorbing barrier. In this section we discuss the properties of the time integral process. The main advantage of this process is that when a realization of the original process ends on the absorbing state, the associated time integral becomes a constant. We can evaluate this constant by again considering small fluctuations around the deterministic motion. Again this approximation works outside the critical region. At the end of Sec. V results obtained in both approximation schemes are compared with the numerical solution of the original stochastic differential equation and the analytic results available at the steady state.

Section VI is devoted to the time integral process properties at the critical point. We show that for this process, as for the original one, scaling behavior appears in the asymptotic regime. A heuristic argument to explain the long-time behavior is presented and, as a consequence, a sum rule for the n-time correlation function and an asymptotic expression for the two-times correlation function are given.

Appendix A gives an alternative derivation of the moments of the integral of a log-normal process. Appendix B gives details of calculations of the generating function of the time-averaged process, and Appendix C shows details of its asymptotic expansion.

II. MODEL

We consider the so-called Malthus-Verhulst stochastic model previously introduced and studied by several authors [8,10,12]. The model is given in terms of a stochastic differential equation (SDE) describing the process evolution:

$$dx = \left[\left[\delta + \frac{\epsilon}{2} \right] x - x^2 \right] dt + \sqrt{\epsilon x}\, dw \ . \tag{2.1}$$

In Eq. (2.1) w is a Wiener process whose increment dw satisfies the properties

$$\langle dw(t) \rangle = 0, \quad \langle dw(t) dw(t') \rangle = \delta(t-t') dt \ . \tag{2.2}$$

We adopt in Eq. (2.1) the Ito prescriptions [15]. In the absence of noise Eq. (2.1) describes the overdamped motion of a particle in the potential shown in Fig. 1(a). By varying the control parameter δ, the particle approaches an equilibrium state ($x_0 = \delta$) for $\delta > 0$ or the ab-

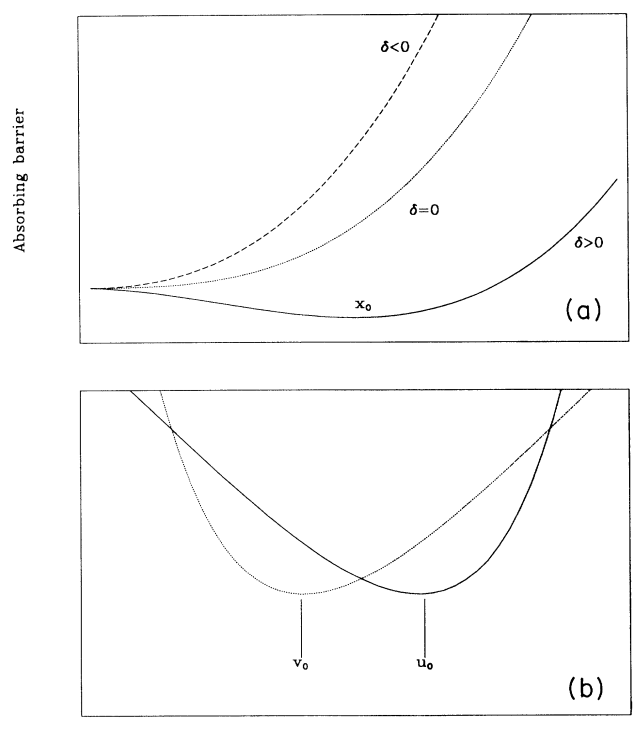

FIG. 1. (a) Schematic view of the potential of the deterministic equation of motion for different value of the parameter δ. (b) Potential $\mathcal{V}$ (solid line) and $\mathcal{W}$ (dotted line) of Eqs. (4.2) and (5.3), respectively, for $\delta > 0$ and $\delta < 0$.

sorbing barrier $x=0$ for $\delta \le 0$.

In the presence of noise steady-state properties have been discussed extensively by several authors and an exhaustive analysis can be found in [12]. A steady-state probability distribution function (PDF) is $\delta(x)$ for $\delta \le 0$ and changes in a regular function as $\delta > 0$. There is another transition marked by a shift in the most probable value of the process from a vanishing value, for $\delta \le \epsilon/2$, to a novanishing value ($x=\delta$), for $\delta > \epsilon/2$. This effect is known as the noise-induced phase transition (NIPT) [10].

Relaxation properties have been also considered in [12], where rather involved formulas for the process moments are derived and discussed. The main result is the scaling behavior derived by Suzuki and co-workers [10] at the critical point.

The solution of Eq. (2.1) is

$$x(t)=x(0)\exp[\delta t+\sqrt{\epsilon}w(t)][1+x(0)z(t)]^{-1} , \quad (2.3)$$

where

$$z(t)=\int_0^t dt' e^{\delta t'+\sqrt{\epsilon}w(t')} . \quad (2.4)$$

We note that the process x is given in terms of two processes: the log-normal process $\exp[\delta t+\sqrt{\epsilon}w(t)]$ and its time integral z.

During the decay toward the absorbing barrier we will study the time integral process Z, which is related simply to the linear process z via the mapping

$$Z(t)=\ln[1+x(0)z(t)] . \quad (2.5)$$

The time integral process reduces to z in the early stages of evolution.

Failure of naive self-consistent approximation

As a preliminary remark let us show the failure, in the case of a multiplicative noise process, of a widely used linearization scheme: the self-consistent approximation.

We consider instead of the original process of Eq. (2.1) a new process obeying the following SDE:

$$dx'=\left[\delta+\frac{\epsilon}{2}-m(t)\right]x'dt+\sqrt{\epsilon}x'dw . \quad (2.6)$$

The variable m must be calculated self-consistently

$$m(t)=\langle x(t)\rangle . \quad (2.7)$$

In Eq. (2.7) the average is taken with respect to the process realization ensemble. From Eqs. (2.6) and (2.7) we obtain a differential equation for the average of the process m:

$$\dot{m}=\left[\delta+\frac{\epsilon}{2}-m\right]m . \quad (2.8)$$

Equation (2.6) can be solved readily in terms of the time integral $M(t)$ of the function m

$$x'(t)=x'(0)\exp[\delta t-M(t)+\sqrt{\epsilon}w(t)] . \quad (2.9)$$

The distribution of x' is a typical log-normal distribution whose moments are dominated, for long times, by rare but large fluctuations

$$\langle x'n\rangle=\langle[x'(0)]^n\rangle\exp\left[n[\delta t-M(t)]+n^2\frac{\epsilon}{2}t\right] . \quad (2.10)$$

Even if the average of the process moves towards a finite equilibrium value $m(t\to\infty)=\delta+\epsilon/2$ for $\delta>0$ or to the absorbing barrier $m(t\to\infty)=0$ for $\delta+\epsilon/2\le 0$, the linearization prevents the elimination of the rare, large fluctuations which give rise to the explosion of large-order moments. In other words, the self-consistent linearization procedure gives results which are qualitatively wrong. This seems to be a peculiar difficulty of the decay in the presence of multiplicative noise.

III. FLUCTUATIONS AMPLIFICATION IN THE SMALL-NOISE APPROXIMATION

Anomalous fluctuations, i.e., nonmonotonic growth of fluctuations during the decay towards the equilibrium state of the process, have been predicted to occur in the presence of the multiplicative noise if the initial state is sufficiently far from the equilibrium state [8].

It is interesting to note that numerical studies of the probability density in the transient evidentiate a bimodal behavior analogous to that observed in the decay of an unstable state in the presence of additive noise [9,16]. To understand the anomalous fluctuations in the context of MVSM we found it convenient to develop a linear analysis of the model.

This is a naive approximation procedure which leads to quantitatively incorrect results, but allows us to introduce the concept of anomalous fluctuations. Direct linearization around $\epsilon=0$, the deterministic evolution x_d of Eq. (2.1), introduces Gaussian fluctuations process $\tilde{x}:x=x_d+\sqrt{\epsilon}\tilde{x}$. Here the evolution of the *deterministic* part x_d is

$$\dot{x}_d=\delta x_d-x_d^2 \quad (3.1)$$

with initial conditions $x_d(0)=x(0)>0$. The fluctuating part evolves according a linear equation

$$d\tilde{x}=\gamma(t)\tilde{x}dt+x_d(t)dw \quad (3.2)$$

with vanishing initial conditions. In Eq. (3.2) γ is given by

$$\gamma(t)=\delta-2x_d(t) . \quad (3.3)$$

We see that, due to fluctuations of Gaussian character, the approximated linear process x is no longer always positive. This is an expected drawback of the direct linearization approximation. The solution of Eq. (3.1) reads

$$x_d=\delta x(0)\{[\delta-x(0)]\exp(-\delta t)+x(0)\}^{-1} . \quad (3.4)$$

From the condition $\gamma>0$ using Eq. (3.4) we easily see that for $\delta>0$ fluctuations are amplified if the initial state is on the left of the equilibrium state $x=\delta$. This happens for times lower than a time t_c given by

$$t_c=\frac{1}{\delta}\ln\left[\frac{\delta}{x(0)}-1\right] \text{ for } \delta>0 \;. \tag{3.5}$$

During this time regime [which can be very large if the initial data $x(0)$ are sufficiently small] the failure of the linearization procedure is expected.

The drawback of the approximation scheme previously introduced can be overcome introducing first a mapping of the original process in a new process given by an additive noise Langevin equation with a single stationary state and then small noise expansion. The approximation will be shown to be successful at least far from the critical region.

IV. DECAY TOWARDS THE EQUILIBRIUM STATE

The small noise expansion is expected to be successful if applied to the *normal* behavior, i.e., to a relaxation regime in which fluctuations are small compared to a suitably chosen deterministic evolution. The new processes are obtained respectively considering the logarithm of the process x for $\delta>0$ and the finite time average of the process x when $\delta<0$.

In the first case we consider the process $u(t)=\ln[x(t)]$. From Eq. (2.1) and using Ito's rules of calculus we obtain

$$du=-\frac{\partial \mathcal{V}(u)}{\partial u}+\sqrt{\epsilon}dw \;. \tag{4.1}$$

Where the potential $\mathcal{V}$, depicted in Fig. 1(b), is given by

$$v(u)=-\delta u+e^{u} \;. \tag{4.2}$$

It is worth noting that the potential v is a single-well potential for $\delta>0$ with a minimum located at $u_0=\ln(\delta)$, which disappears as $\delta\geq 0$. The normal behavior of this process is due to the appearance of the single-well potential and of an additive noise.

We separate the process u into a deterministic part u_d and a small fluctuating part $\tilde{u}$ the equation for the deterministic part reads

$$\dot{u}_d=f(u_d) \;, \tag{4.3}$$

where the deterministic drift is

$$f(u)=\delta-e^{u} \;, \tag{4.4}$$

while the equation for the fluctuating part reads

$$d\tilde{u}=-e^{u_d}\tilde{u}dt+\sqrt{\epsilon}dw \tag{4.5}$$

with a vanishing initial condition.

The equation for u_d can be easily solved giving

$$u_d(t)=\ln\left[\frac{\delta x(0)}{[\delta-x(0)]\exp(-\delta t)+x(0)}\right] \;. \tag{4.6}$$

From Eq. (4.5) we easily obtain the variance $\sigma_u^2(t)=\langle\tilde{u}^2\rangle$ of the process $\tilde{u}$

$$\sigma_u^2=\epsilon f^2(u_d)\int_{u(0)}^{u_d(t)}\frac{dx}{f^3(x)} \;. \tag{4.7}$$

Using the solution of Eq. (4.6) and the definition (4.4) we obtain

$$\sigma_u^2=\frac{\epsilon}{2\delta}\left[1-\frac{f(t)^2}{f(0)^2}\right]+\frac{\epsilon f(t)}{\delta^2}\left[1-\frac{f(t)}{f(0)}\right]+\frac{\epsilon t}{\delta^2}f(t)^2 \;, \tag{4.8}$$

where we have used the shorter notation $f(t)=f(u_d(t))$. Since the asymptotic fluctuations variance is $\sigma_u^2(\infty)=\epsilon/2\delta$, we have that linear fluctuations expansion is valid asymptotically if $\epsilon\ll 2\delta$. From the dynamical point of view we see that the restoring force of the linear fluctuations evolution vanishes as δ becomes of the order of the noise variance ϵ.

The time evolution of the variance σ_u^2 together with the knowledge of the deterministic evolution u_d completely determine the single time properties of the original process x. In particular we notice that the distribution function of the process is log-normal.

$$P(x,t)=\{x[2\pi\sigma_u^2(t)]^{1/2}\}^{-1}\exp\left[-\frac{[\ln(x)-u_d(t)]^2}{2\sigma_u^2(t)}\right] \;. \tag{4.9}$$

The associated moments are easily given in terms of the same quantities:

$$\langle x^n(t)\rangle=\exp\left[nu_d(t)+\frac{n^2\sigma_u^2(t)}{2}\right] \;. \tag{4.10}$$

From Eq. (4.10) it is possible to evaluate the fluctuations of the process $\langle\Delta x^2\rangle=\langle x^2\rangle-\langle x\rangle^2$ in the transient. Analytical results obtained in the present approximation scheme are compared with numerical solution of the Langevin equation (2.1) in Fig. 2.

We have anomalous behavior of fluctuations in the transient in two cases. In case A of Fig. 2, in which the initial condition is close the absorbing barrier, the enhancement of fluctuations is related to the random departure of the process from a nearly unstable state under the action of a very small noise. This case is similar to the decay from an initial unstable state triggered by an additive noise [9,16]. An important difference is that in this case the amplitude of the fluctuations' peak increases with the noise strength while in the additive noise case it is almost constant. This is due to the relative importance of the noise in the whole transient and not only in the early stage of growth.

In case B of Fig. 2 the initial state is far away from the absorbing barrier. Fluctuations initially grow due to the large noise and then they decrease as the process relaxes towards states affected by a smaller noise.

This difference is even more evident in the normalized fluctuations $\langle\Delta x^2\rangle/\langle x\rangle^2$ (see Fig. 3). This quantity shows an anomalous behavior only in case A. From Eq. (4.10) we see that if $\sigma_u^2\ll 1$, then σ_u^2 approximates the normalized fluctuations of the process x. The behavior of the normalized fluctuations is depicted in Fig. 3(a) for two different choices of initial data: close and far away from the absorbing barrier. We note a substantially

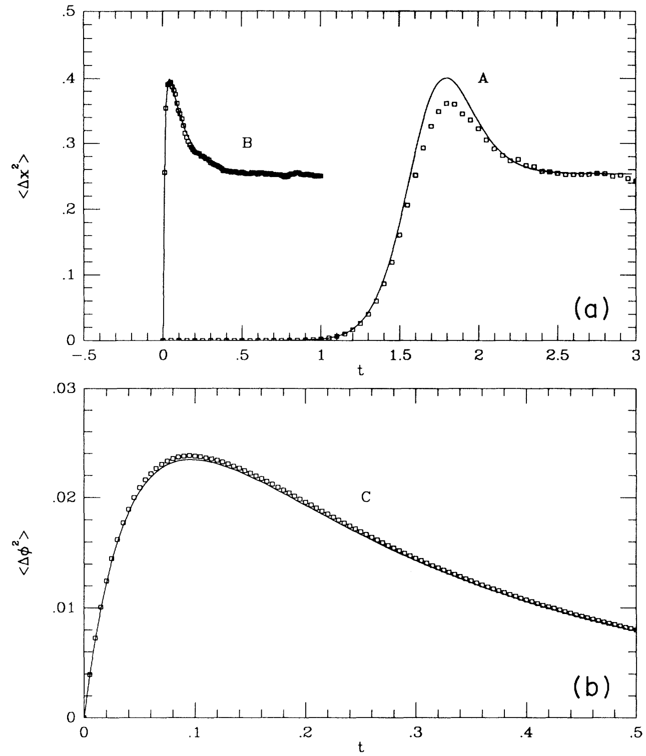

FIG. 2. (a) Fluctuation of the process starting with two different initial conditions A and B. (b) Fluctuations of the time average of the process starting with initial conditions C. The solid line is the result of the theory. Squares are the results of numerical integration of the Langevin equation (2.1). As in Figs. 3, 4, and 6, data are obtained starting with the following parameter settings: A, $\delta=5>0$ and $x(0)=10^{-4}\ll x_0$; B, $\delta=5>0$ and $x(0)=20\gg x_0$; C, $\delta=-5<0$ and $x(0)=5$. In all conditions $\epsilon=0.1$.

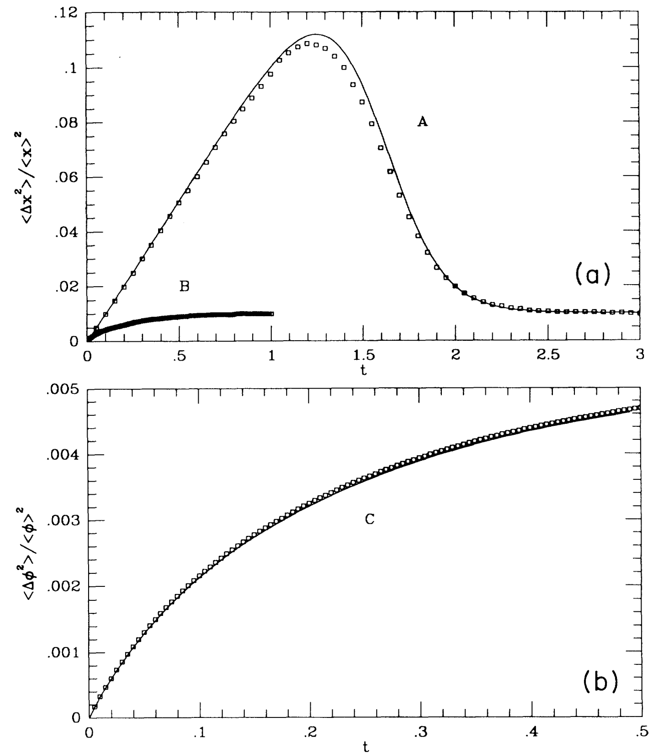

FIG. 3. (a) Normalized fluctuations of the process x. (b) Normalized fluctuations of the time-averaged process ϕ.

different behavior and consequently we expect that the variance evolution largely depends on the initial state $x(0)$. We observe a nonmonotonic behavior of the normalized fluctuations if the process is initially close to the absorbing barrier. If the process starts on the left of the equilibrium point of the potential $\mathcal{V}(u)$ [see from Fig. 1(b)] is driven away from the absorbing barrier by a constant drift δ. Thus initially fluctuations increase linearly in time as ϵt. This increase stops once the process meets the sharp exponential branch of the potential. This happens roughly in a time scale t_s given by $t_s \simeq [u_{\rm eq}-u(0)]/\delta$. At the same time fluctuations are of the order

$$\sigma_u^2 \simeq \frac{\epsilon}{\delta}\ln\left[\frac{\delta}{x(0)}\right], \tag{4.11}$$

which can be much larger than the asymptotic value if the initial position is sufficiently close the absorbing barrier at $x=0$.

This fact may affect the accuracy of our approximation in the time regime in which the variance reaches the maximum value predicted in Eq. (4.11). We can see this phenomenon by comparing our prediction with the numerical simulation of Eq. (2.1). For example, in the transient behavior of the process fluctuations [Fig. 2(a)] we observe discrepancies only for the initial states close to the absorbing barrier. For the same choice of initial value of the process it can be noted from Fig. 4(a) that our approximation overestimates realizations far from the absorbing barrier during the transient.

From Eq. (4.9) and from the asymptotic value of the variance it is possible to compare our log-normal PDF, evaluated asymptotically with the known steady-state analytical results [12] [see Fig. 5(a)]. The NIPT associated with the disappearance of the peak in the steady-state PDF as $\delta<\epsilon/2$ is not observed in our approximation scheme in which PDF is always log-normal and consequently has always a nonvanishing peak. However, our approximation outside the critical region is in reasonable agreement with known steady-state properties.

Finally it is important to realize that our approximations is a *process approximation* which gives information about single trajectories (or process realizations) and not only about the averaged properties of PDF. In Fig. 6(a) the approximate process obtained by numerically solving the Langevin equation (4.5) and using Eqs. (4.6) and (4.7) compared with numerical solution of the model equations for a given realization is shown.

V. DECAY TOWARDS THE ABSORBING BARRIER

We consider now the $\delta<0$ case, i.e., the parameter's space region where the system decays towards the ab-

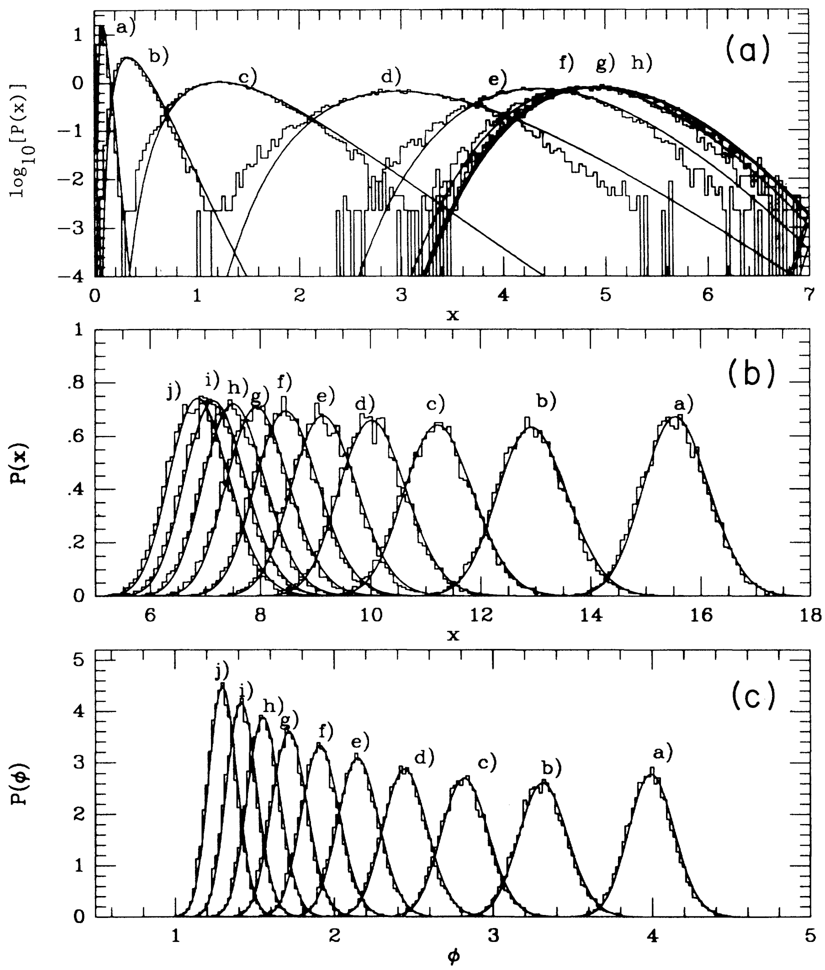

FIG. 4. (a) PDF of the process x starting with initial condition A sampled at eight different times from (a) $t=0.9$ to (h) $t=3.0$ with time step 0.3. The solid line is the theory. Histograms are obtained by averaging 12 800 realizations of the process x. (b) The same, but with initial condition B. Sampling times run from (a) $t=0.02$ to (j) $t=0.2$ with time step 0.02. (c) PDF of the time-averaged process ϕ. Sampling times runs from (a) $t=0.05$ to (j) $t=0.5$ with time step 0.05.

sorbing barrier. The mapping used in the previous case is now useless because the restoring force acting on the process fluctuations $\tilde{u}$ vanishes asymptotically as the deterministic motion approaches the absorbing barrier. In order to overcome this difficulty we propose to study the time integral of the original process x introduced in Eq. (2.5). There is a simple consideration which justifies the choice of the time integrated process in order to introduce simple approximation schemes. The time integral of the process approaches a constant value as the process itself approaches the absorbing barrier. We can distinguish the deterministic time integral process and the fluctuations time integral process. Since the former is a nonvanishing process, the latter can be expected to be a small perturbation.

Using the translation invariance properties of the Wiener process it is easy to show that the process z satisfies the following SDE:

$$dz=\left[\left[\delta+\frac{\epsilon}{2}\right]z+1\right]dt+\sqrt{\epsilon}z\,dw\ . \tag{5.1}$$

It is again convenient to study the normally behaved process $V=\ln(z)$ which obeys an additive noise stochastic equation easily derived from Eq. (5.1):

$$dv=-\frac{d\mathcal{W}}{dv}dt+\sqrt{\epsilon}\,dw \tag{5.2}$$

where the potential $\mathcal{W}$, depicted in Fig. 1(b), is given by

$$\mathcal{W}=-\delta v+e^{-v}\ . \tag{5.3}$$

The potential $\mathcal{W}$ is a single-well potential for negative δ with a minimum located at $v_0=-\ln(|\delta|)$. The probabilistic evolution in the presence of the potential $\mathcal{W}$ can be studied by means of the Fokker-Planck equation general methods of Ref. [17]. The particular case $\delta=\epsilon/2-1$ corresponds to the well-known case of Morse potential [18]. The general case has been discussed in [19]. An alternative approach, with results more effective in the study of the transient behavior, is the following.

We introduce again a linear approximation scheme for the process v separating it into a deterministic part v_d and a small fluctuating part $\tilde{v}$. The deterministic evolution is given by

$$v_d(t)=\ln\left[\frac{1}{\delta}(e^{\delta t}-1)\right]\ . \tag{5.4}$$

The initial condition $v_d(0)=-\infty$ is a consequence of the vanishing initial value of the time integral process z.

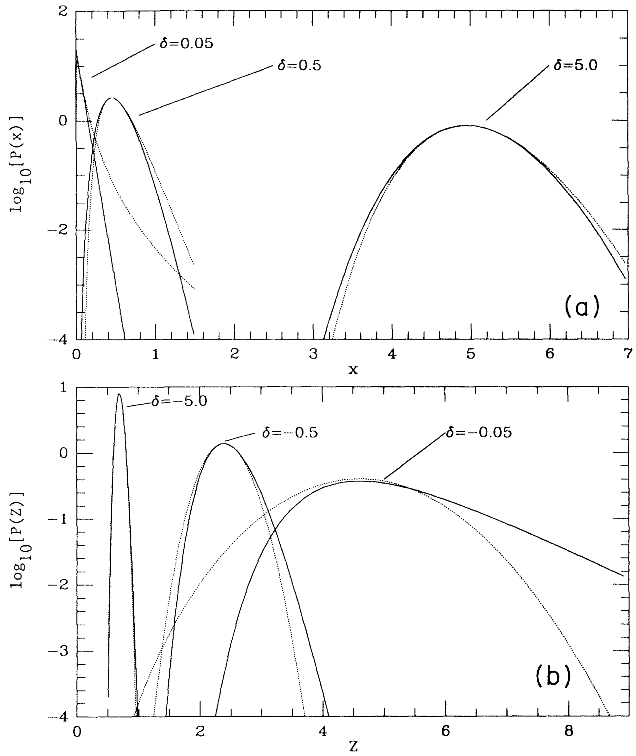

FIG. 5. (a) Exact asymptotic PDF [12] (solid lines) compared with that obtained by our approximation (dotted lines) with several values of the parameter δ. (b) Exact asymptotic PDF of the time integral process Z given by Eq. (5.12) (solid lines) compared with our predictions (dotted lines).

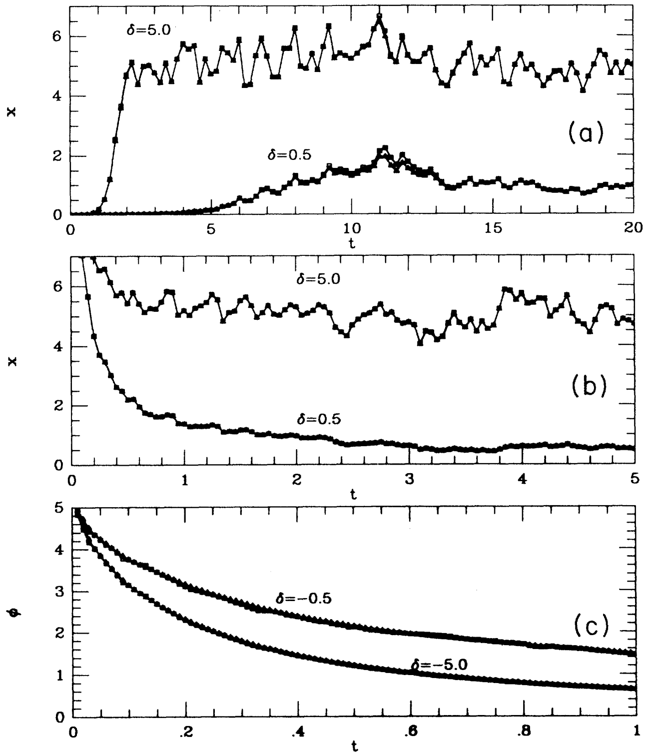

FIG. 6. Several realizations of the process starting from conditions (a) A and (b) B (triangles) compared with our approximated process (squares). (c) The same for the time-averaged process ϕ with initial conditions C. Simulations of the original and approximated processes are obtained numerically, taking the same series of pseudo-random numbers to generate the Wiener processes involved in their definitions.

The fluctuation process $\tilde{v}$ obeys the following SDE:

$$d\tilde{v} = -e^{v_d(t)}\tilde{v}dt + \sqrt{\epsilon}dw \ . \qquad (5.5)$$

We see from Eq. (5.5) that for v_d varying from $-\infty$ to v_0 the restoring force is always negative and no fluctuation amplifications occur. As expected, the restoring force decreases as $\delta \to 0$. In this limit we enter a critical region of the parameters space.

The explicit calculation of the variance of the fluctuating part $\tilde{v}$ gives

$$\sigma_v^2(t) = -\frac{\epsilon}{2\delta} - \frac{\epsilon}{\delta^2}f(t) + \frac{\epsilon t}{\delta^2}f(t)^2 \ . \qquad (5.6)$$

As in Sec. V, f is the drift of the deterministic evolution evaluated in $v_d(t)$

$$f(t) = \frac{\delta}{1-e^{-\delta t}} \ . \qquad (5.7)$$

We see that, for negative δ, σ_v^2 approaches monotonically the asymptotic value $\sigma_v^2(\infty) = \epsilon/2|\delta|$. Again the validity of the previous calculations is restricted by the requirement $\sigma_v^2 \ll 1$, which gives $\epsilon \ll 2|\delta|$.

As far as the moments of the time-averaged process are concerned, they are given in terms of averages over a Gaussian variable v with mean v_d and variance σ_v. By using the mapping given in Eq. (2.5) we can write the PDF of the process Z at any time as

$$P(Z,t) = \{(1-e^{-Z})[2\pi\sigma_v^2(t)]^{1/2}\}^{-1} \times \exp\left[-\frac{\{\ln[(e^Z-1)/x(0)] - v_d(t)\}^2}{2\sigma_v^2(t)}\right] \ . \qquad (5.8)$$

To compare our analytical results with numerical simulation we introduce the time average of the process x: $\phi(t) = Z(t)/t$. This quantity goes to zero like the moments of the original process as time increases. By using the definition of ϕ and the PDF of Eq. (5.8) it is easy to obtain the expression for the moments of the time-averaged process ϕ,

$$\langle \phi(t)^n \rangle = \frac{1}{t^n}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2x}} e^{-x^2/2} \ln^n[1 + x(0)e^{v_d(t)+\sigma_n(t)x}] \ . \qquad (5.9)$$

We checked the approximation for the moments of time average, comparing results obtained for fluctuations $\langle \Delta\phi^2 \rangle = \langle \phi^2 \rangle - \langle \phi \rangle^2$ [see Fig. 2(b)] and for the normalized fluctuations $\langle \Delta\phi^2 \rangle / \langle \phi \rangle^2$ [see Fig. 3(b)] with numerical solution of the model equation (2.1). Unlike the $\delta > 0$ case, normalized fluctuations do not exhibit anomalous behavior. This is simply due to the fact that the process

x experiences monotonically decreasing noise. Transient and steady-state PDF are shown in Figs. 4(c) and 5(b), respectively. The steady-state PDF is compared with exact result obtained by analyzing the steady-state solution P_0 of Fokker-Planck equation associated with the Langevin equation [Eq. (5.1)] for the process z. The equation for P_0 reads

$$\frac{\epsilon}{2}\frac{\partial}{\partial z}z^2P_0-\left[\delta+\frac{\epsilon}{2}\right]zP_0-P_0=0 . \quad (5.10)$$

The solution of this equation is

$$P_0(2)=\mathcal{N}^{-1}z^{2\delta/\epsilon-1}\exp(-2/\epsilon z) , \quad (5.11)$$

where the normalization constant is $\mathcal{N}=\Gamma(-2\delta/\epsilon)(2/\epsilon)^{2\delta/\epsilon}$. We note that the distribution is normalizable, i.e., the process approaches a steady state, only if $\delta<0$. Using Eqs. (2.5) and (5.11) it is easy to write the asymptotic PDF for the time integral process Z:

$$P_{SS}(Z)=\mathcal{N}\left[\frac{x(0)}{e^Z-1}\right]^{1-2\delta/\epsilon}\exp\left[-\frac{2x(0)}{\epsilon(e^Z-1)}\right] . \quad (5.12)$$

VI. ASYMPTOTIC BEHAVIOR AT THE CRITICAL POINT

One of the most fundamental features of critical dynamics is the phenomenon of critical slowing down, which means that the relaxation of the moments of the stochastic process becomes very slow near the critical point. Suzuki and co-workers in fact have shown that all the moments of the original process of Eq. (2.1) have the same asymptotic behavior as $t^{-1/2}$ at the critical point. Consequently the same long-time law for the time average is expected.

It is also interesting to determine whether higher moments of the time average have the same long-time tails. It is difficult to deduce the long-time behavior of the time average moments from the properties known from the original process because the expression of the nth moment of the time average involves an n-times correlation function of the original process. On the other hand, the independent knowledge of the properties of the time-averaged process helps us to determine some feature of the correlation functions of the original process.

Analyzing the expression (2.5), which relates the time-averaged process to the linear process z, it is possible to derive the long-time behavior by means of an exact asymptotic expansion of the time averaged process generating function. The result will be derived independently with a heuristic argument.

The generating function for the time-averaged moments is defined as

$$G_\phi(\lambda,t)=\langle\exp(-\lambda\phi)\rangle . \quad (6.1)$$

From Eq. (2.5) we have

$$g_\phi(\lambda,t)=\langle[1+x(0)z(t)]^{-q}\rangle , \quad (6.2)$$

where $q=\lambda/t$. Using the identity

$$(1+a)^b=\frac{\int_0^\infty e^{-(a+1)\mu}\mu^{-b-1}d\mu}{\int_0^\infty e^{-\mu}\mu^{-b-1}d\mu} , \quad (6.3)$$

which holds for any value of a, we have, after averaging,

$$G_\phi(\lambda,t)=\frac{\int_0^\infty G_z(\mu,t)e^{-\mu}\mu^{q-1}d\mu}{\int_0^\infty e^{-\mu}\mu^{q-1}d\mu} . \quad (6.4)$$

If we consider t as a complex variable in the left half of the complex plane, we can use the expression of the generating function given in Eq. (6.4). We then obtain

$$G_\phi(\lambda,t)=\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(q+k)}{k!\Gamma(q)}x(0)^k\langle z^k(t)\rangle . \quad (6.5)$$

The moments of the process z can be derived as an application of the general method of Ref. [20] and then summed up to obtain the following expression of the generating function at the critical point (details are given in Appendixes A and B):

$$G_\phi(\lambda,t)=2\frac{\sin(\pi q)}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\, e^{-x^2/2}\int_0^1du\, e^{\beta u}(1-u)^{-q}u^{q-1}\,{}_1F_1(1,1-q,y)\,{}_{-1}F_1(q;1,\beta) . \quad (6.6)$$

The expression Eq. (6.6) has a form that is easy to handle to get the asymptotic expansion of the generating function and consequently of the PDF and the moments of the time-averaged process. By means of a rather lengthy calculation summarized in Appendix C we obtain asymptotically

$$G_\phi(\lambda,t)=2e^{\epsilon\lambda^2/2t}\text{erfc}(\lambda\sqrt{\epsilon/2t}) . \quad (6.7)$$

The final result of Eq. (6.7) can also be obtained with the following heuristic argument.

Let us suppose that $x(0)z(t)\gg 1$ in Eq. (2.5). It is easy to observe that the integral that defines the linear process z [Eq. (2.4)] can be approximated by

$$z(t)\simeq Z(t)\exp[\sqrt{\epsilon}w_{\max}(t)] , \quad (6.8)$$

where $w_{\max}(t)=\sup_{0<t'<t}w(t')$ and Z is a function which diverges les than a linear function of time. As a consequence the time-averaged process will be proportional to the process $w_{\max}$. The distribution of this process is known [15]. We can write

$$\phi(t)\simeq\frac{m}{\sqrt{t}} , \quad (6.9)$$

where m is a random variable distributed according a *semi-Gaussian* distribution

$$P(m)=\begin{cases}2(2\pi\epsilon)^{-1/2}\exp(-m^2/2\epsilon) & \text{for } m\geq 0\\ 0 \quad \text{for } m<0 .\end{cases} \quad (6.10)$$

The moments of the time average are readily evaluated

$$\langle \phi^n \rangle \simeq \frac{\mu_n}{t^{n/2}} , \tag{6.11}$$

where μ_n is the nth moments of the distribution of Eq. (6.10)

$$\mu_n = 2\epsilon(\pi)^{-1/2}\Gamma\left[\frac{n+1}{2}\right] . \tag{6.12}$$

These moments coincide with the moments associated with the generating function of Eq. (6.7). The long-time behavior of the time-averaged moments is in agreement with numerical simulation of Eq. (2.1) (see Fig. 7).

It is important to notice that this result implies an asymptotic *sum rule* for the n-times correlation function of the original process

$$\lim_{t\to\infty} \frac{1}{t^{n/2}} \int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n \langle x(t_1)x(t_2)\cdots x(t_n)\rangle = \mu_n . \tag{6.13}$$

This sum rule is consistent with the hypotesis that correlations have power-law tails. For example, let us consider the usual two-times correlation function

$$C(t_1,t_2) = \langle x(t_1)x(t_2)\rangle . \tag{6.14}$$

We can rewrite the correlation function as

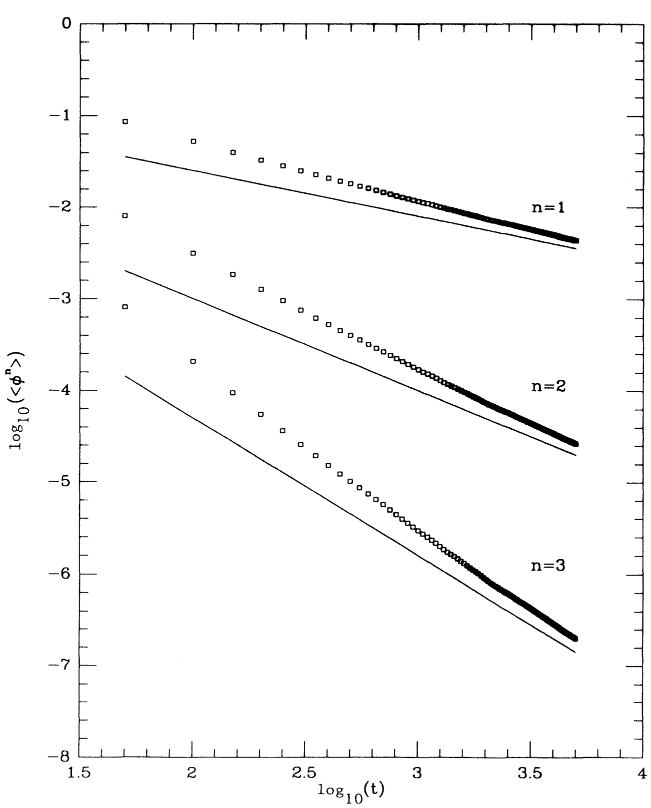

FIG. 7. Long-time behavior of the moments ϕ^n (squares) of the time-averaged process at the critical point $\delta=0$ compared with asymptotic expansion of Eqs. (6.11) and (6.12) (solid line). The average is performed on 1536 runs.

$$C(t_1,t_2) = \langle \langle x(t_1)\rangle|_{x(t_2)} x(t_2)\rangle \tag{6.15}$$

where $\langle\ \rangle|_{x(t_2)}$ means a constrained average on realizations of x which starting from $x(0)=x_0$ pass by $x(t_2)$ when $t=t_2$. The solution of Eq. (2.1) with the constraint $x(t_2)=x_2$ is readily obtained from Eq. (2.3) once $t=0$ is replaced with $t=t_2$. For times $t_1 \gg t_2$ the process is independent of the "initial data" x_2, but still depends on both times t_1 and t_2

$$x(t_1) \simeq \frac{e^{w(t_1-t_2)}}{\int_{t_2}^{t_1} dt' e^{w(t')}} . \tag{6.16}$$

Using Eq. (6.16) together with Suzuki and co-workers' asymptotic result $\langle x^n \rangle \simeq a_n/t^{1/2}$ we get

$$C(t_1,t_2) \simeq a_1^2[t_2(t_1-t_2)]^{-1/2} . \tag{6.17}$$

We note therefore that the original process is not stationary. Integrating this formula for $n=2$ using Eq. (6.13) we obtain

$$\mu_2 = 2a_1^2\pi , \tag{6.18}$$

where the factor 2 is due to time ordering. The coefficient a_1 can be obtained directly by differentiating Eq. (6.11),

$$\frac{\mu_2}{\mu_1^2} = \frac{\pi}{2} . \tag{6.19}$$

It is easy to check that this formula is consistent with expression μ_n obtained using Eq. (6.10).

We want to emphasize that this heuristic procedure is an alternative derivation of the results of Suzuki and coworkers for the moments of the original process. Finally we recall that the long-time tails in the original process are associated with the persistence on a large but finite time scale of a macroscopic number of realizations which did not collapse yet onto the absorbing barrier.

VII. CONCLUSIONS

The main purpose of the present study was to show how the transient statistical properties of a process in the presence of a multiplicative noise can be approximated within a standard small-noise expansion technique. In the case of the decay away from the absorbing barrier it is necessary to find a suitable mapping which introduces to the original process a process moving in a single-well potential under the action of additive noise. In the case of decay towards the absorbing barrier we propose to develop the approximation scheme to the finite time-averaged process. The latter process is also suitable for the study of the critical point fluctuations.

This work naturally leads to the consideration of possible experiments designed to detect statistical properties of the time-averaged process in the transient regime. The possible extension to a multidimensional process can be of interest in the study of transient behavior of multimode laser as well as the ecological model of interacting populations.

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APPENDIX A: MOMENTS OF THE PROCESS Z

To derive a differential equation for the moments of the process z defined in Eq. (2.4), we follow the method of Ref. [20]. We discretize the time interval $[0,t]$ into N steps of width Δt. The process z at the $(N+1)$th step is given by the recursion

$$Z_{N+1}=\Delta t+z_N\exp(\delta\Delta t+\sqrt{\epsilon}\Delta w_N)\ , \tag{A1}$$

where Δw_N is the increment of a discretized Wiener process, i.e., a zero mean Gaussian uncorrelated variable

$$\langle \Delta w_k\rangle=0,\quad \langle \Delta w_k \Delta w_l\rangle=\delta_{k,l}\Delta t\ . \tag{A2}$$

From Eq. (A1) we get

$$\langle (z_N+1-\Delta t)^n\rangle=\langle z_N^n\rangle\exp(\alpha_n\Delta t)\ , \tag{A3}$$

where

$$\alpha_n=n\delta+n^2\frac{\epsilon}{2}\ . \tag{A4}$$

In Eq. (A3) we have taken into account the statistical independence, in the Ito scheme, of z_N from the increment of the Wiener process Δw_N. Taking the limit of small Δt we get a set of differential equations for the moments m_n, of the process z,

$$\frac{dm_n}{dt}=\alpha_n m_n+nm_{n-1}\ . \tag{A5}$$

Equation (A5) must be solved with initial conditions

$$m_n(0)=1 \quad \text{for } n\geq 0 \tag{A6}$$

and

$$m_0(t)=1 \quad \text{for any time}\ . \tag{A7}$$

By means of Laplace transform Eq. (A5) becomes the recursion relations

$$m_n(s)=\frac{nm_{n-1}(s)}{s-\alpha_n} \quad \text{for } n>0\ , \tag{A8}$$

with

$$m_0(s)=\frac{1}{s}\ . \tag{A9}$$

The solution reads

$$m_n(s)=\frac{A_0^{(n)}}{s}+\sum_{k=1}^{n}\frac{A_k^{(n)}}{s-\alpha_k}\ , \tag{A10}$$

where

$$A_0^{(n)}=\frac{(-1)^n}{\prod_{k=1}^{n}(\delta+\epsilon k/2)}\ ,$$
$$A_k^{(n)}=(-)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix}\frac{\delta+\epsilon k}{\prod_{l=0}^{n}[\delta+\epsilon(k+l)/2]}\ . \tag{A11}$$

We notice that at the critical point $\delta=0$ in Eq. (A11) the limit $k\to 0$ does not give $A_0^{(n)}$.

APPENDIX B: THE GENERATING FUNCTION OF PROCESS ϕ

By the inverse transformation from Eq. (A10) we obtain

$$\langle z^k(t)\rangle=A_0^{(k)}+\sum_{l=1}^{k}A_l^{(k)}\exp(\alpha_l t)\ , \tag{B1}$$

where the coefficients of Eqs. (A11) at the critical point read

$$A_0^{(k)}=\left[-\frac{2}{\epsilon}\right]^k\frac{1}{k!}\ ,$$
$$A_l^{(k)}=(-1)^{k-l}\left[\frac{2}{\epsilon}\right]^k\frac{2k!}{(k-l)!(k+l)!}\ , \tag{B2}$$

and $\alpha_k=\epsilon k^2/2$. Substituting Eq. (B1) into Eq. (6.5) we get

$$G=G^{(a)}+G^{(b)}\ , \tag{B3}$$

where

$$G^{(a)}=\sum_{k=0}^{\infty}\frac{\Gamma(k+q)}{k!\Gamma(q)}\frac{\beta^k}{k!}\ , \tag{B4}$$

$$G^{(b)}=\sum_{k=1}^{\infty}\frac{\Gamma(k+q)}{k!\Gamma(q)}\beta^k\sum_{l=0}^{k}(-1)^l\frac{2k!}{(k-l)!(k+l)!}e^{\alpha_l t}\ , \tag{B5}$$

and $\beta=2x(0)/\epsilon$.

To obtain a meaningful analytic continuation of the generating function in right half-plane we must perform a resummation of the series appearing in Eqs. (B4) and (B5). The first term appearing in Eq. (B3) can be expressed as a confluent hypergeometric function [21]

$$G^{(a)}(\lambda,t)={}_1F_1(q;1,\beta)\ . \tag{B6}$$

The second term can be arranged in a more convenient form by the substitution $\sum_{k=1}^{\infty}\sum_{l=1}^{k}\to\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}$ (see, for example, Suzuki and co-workers [10]). Then we can express the sum over k as an hypergeometric function

$$G^{(b)}=2\sum_{l=1}^{\infty}(-\beta)^l e^{\alpha_l t}\frac{\Gamma(l+q)}{(2l)!\Gamma(q)}\,{}_1F_1(l+q;2l+1,2\beta)\ . \tag{B7}$$

Introducing the integral representation of the hypergeometric function

$$ {}_1F_1(a;c,z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\, e^{zu}u^{a-1}(1-u)^{c-a-1} \quad \text{(B8)}$$

and of the exponential appearing in Eq. (B7)

$$ \exp(\alpha_l t)=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\exp(-x^2/2+l\sqrt{\epsilon t}\,x)\ , \quad \text{(B9)}$$

it is possible to rearrange Eq. (B7) as a double integral [22]

$$ G^{(b)}=-\frac{2\beta}{\Gamma(q)\Gamma(2-q)}\times\int_{-\infty}^{\infty}dx e^{-x^2/2}\int_0^1 du\, e^{\beta u}(1-u)^{-q}u^{q-1}\times {}_1F_1(1,1-q,y)\ , \quad \text{(B10)}$$

where $y=-\beta u(1-u)\exp(x\sqrt{\epsilon t}\,)$. By using the relations

$$ \frac{1}{\Gamma(q)\Gamma(1-q)}=\frac{\sin(\pi q)}{\pi} \quad \text{(B11)}$$

and

$$ {}_1F_1(1;1-q,y)=y\frac{\Gamma(1-q)}{\Gamma(2-q)}\,{}_1F_1(1;2-q,y)\ , \quad \text{(B12)}$$

we finally get

$$ G^{(b)}=2\frac{\sin(\pi q)}{\pi}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2}\times\int_0^1 du\, e^{\beta u}(1-u)^{-q}u^{q-1}\times[{}_1F_1(1,1-q,y)-1]\ . \quad \text{(B13)}$$

The second term of Eq. (B13) and $G^{(a)}$ can be collected together to obtain the complete expression of the generating function of the time-averaged process

$$ G_\phi(\lambda,t)=2\frac{\sin(\pi q)}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\, e^{-x^2/2}\int_0^1 du\, e^{\beta u}(1-u)^{-q}u^{q-1}\,{}_1F_1(1,1-q,y)-{}_1F_1(q;1,\beta)\ . \quad \text{(B14)}$$

APPENDIX C: ASYMPTOTIC EXPANSION OF THE GENERATING FUNCTION

Let us call the first and the second terms of Eq. (B14) $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. We will now concentrate our attention on the asymptotic expansion of $\mathcal{J}_1$. We use different expansions for the hypergeometric function with respect to its third variable y

$$ {}_1F_1^{(-)}=\sum_{k=1}^{\infty}f_k^{(-)}y^k \quad \text{for } |y|<1\ , \quad \text{(C1)}$$

$$ {}_1F_1=\sum_{k=1}^{\infty}f_k^{(+)}y^{-k} \quad \text{for } |y|>1\ . \quad \text{(C2)}$$

The explicit form of the coefficients $f^{(+,-)}$ can be found in Ref. [23], in the following we will use only the fact that $f_k^{(-)}=\Gamma(q+k)/(\Gamma(q)k!)$. To use the different expansions of Eqs. (C2) we split the integration domains in Eq. (B14) by introducing the cutoff $a=\ln(4/\beta)/\sqrt{\epsilon t}$ and the solutions of the equation $|y|=1$

$$ u_{1,2}=\tfrac{1}{2}[1\pm(1-4e^{x\sqrt{\epsilon t}}/\beta)^{1/2}]\ . \quad \text{(C3)}$$

In terms of these quantities it is possible to rearrange $\mathcal{J}_1$ as a sum of four contributions

$$ \mathcal{J}_1=2\frac{\sin(\pi q)}{\pi}(\mathcal{J}_1^{(a)}+\mathcal{J}_1^{(b)}+\mathcal{J}_1^{(c)}+\mathcal{J}_1^{(d)})\ , \quad \text{(C4)}$$

where

$$ \mathcal{J}_1^{(a)}=\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\int_0^{u_1}du\, g(u)\,{}_1F_1^{(-)}\ , \quad \text{(C5)}$$

$$ \mathcal{J}_1^{(b)}=\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\int_{u_2}^{1}du\, g(u)\,{}_1F_1^{(-)}\ , \quad \text{(C6)}$$

$$ \mathcal{J}_1^{(c)}=\int_{-\infty}^{a}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\int_0^{1}du\, g(u)\,{}_1F_1^{(-)}\ , \quad \text{(C7)}$$

$$ \mathcal{J}_1^{(d)}=\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\int_{u_1}^{u_2}du\, g(u)\,{}_1F_1^{(+)}\ , \quad \text{(C8)}$$

and $g(u)=e^{\beta u}(1-u)^q u^{q-1}$.

Let us consider $\mathcal{J}_1^{(a)}$ and $\mathcal{J}_1^{(b)}$. Performing the integration in u we get

$$ \mathcal{J}_1^{(a)}=\sum_{k=0}^{\infty}f_k^{(-)}\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2+kx\sqrt{\epsilon t}}\beta^k\frac{u_1^{q+k}}{q+k}\ , \quad \text{(C9)}$$

$$ \mathcal{J}_1^{(b)}=\sum_{k=0}^{\infty}f_k^{(-)}\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{\beta}\beta^k\frac{(1-u_2)^{k-q+1}}{k-q+1}\ . \quad \text{(C10)}$$

Now we consider $e^{-x\sqrt{\epsilon t}}$ as a small quantity. This is correct for large times only if $a>0$, i.e., if $2\epsilon>x(0)$. In the following we shall prove that within the limitations, the results do not depend on the cutoff a. For large times we get

$$ \mathcal{J}_1^{(a)}=\sum_{k=0}^{\infty}f_k^{(-)}\frac{\beta^q}{q+k}\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2-q\sqrt{\epsilon t}} \quad \text{(C11)}$$

and

$$ \mathcal{J}_1^{(b)}=\sum_{k=0}^{\infty}f_k^{(-)}\frac{\beta^{1-q}}{k-q+1}e^{\beta}\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2-(1-q)\sqrt{\epsilon t}}\ . \quad \text{(C12)}$$

Now let us consider $\mathcal{J}_1^{(c)}$. We can perform the integration in u by the series

$$\int_0^1 du\, e^{\beta u}\frac{u^{q+k-1}}{(1-u)^{q+k}} = \sum_{m,n=0}^{\infty}\begin{bmatrix} q-k \\ n \end{bmatrix}\frac{\beta^m}{(m+n+q+k)m!} . \quad (C13)$$

As $q \to 0$ the term with $m=n=k=0$ diverges. Using the fact that $f_0^{(-)}=1$ we get

$$\mathcal{J}_1^{(c)}=\frac{1}{q}\int_{-\infty}^{a}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}+O(1) . \quad (C14)$$

The last term of $\mathcal{J}_1$ reads explicitly

$$\mathcal{J}_1^{(d)}=\sum_{k=1}^{\infty} f_k^{(+)}\int_a^{\infty}\frac{dx}{\sqrt{2\pi}}\frac{\beta}{(k-q+1)(q-k)} \times e^{-x^2/2+\beta-x\sqrt{\epsilon t}} . \quad (C15)$$

We notice that each term of the above series can be integrated with Gaussian measure for vanishing q.

The second term of Eq. (B14) can be evaluated using Eq. (C2) with $y=\beta$ retaining only the zeroth-order term for vanishing q.

We are now able to collect the partial results of Eqs. (C11), (C12), (C14), and (C15) taking into account the prefactor of Eq. (C4).

In the limit of small q we obtain

$$G_\phi(\lambda,t)=2e^{\epsilon\lambda^2/2t}\operatorname{erfc}(a/\sqrt{2}+\lambda\sqrt{\epsilon/2t}) , \quad (C16)$$

where

$$\operatorname{erfc}(z)=\int_z^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2} \quad (C17)$$

and a is a cutoff parameter. In Eq. (C16) we have written the results of the asymptotic expansion of the three relevant terms $\mathcal{J}_1^{(a)}, \mathcal{J}_1^{(c)}$ and $\mathcal{J}_2$, respectively. We have retained terms which can be resummed as a function of $\lambda/\sqrt{t}$ and neglected as a function of λ/t. We can check the influence of the cutoff a in Eq. (C16) by using

$$\operatorname{erfc}(a/\sqrt{2}+\lambda\sqrt{\epsilon/2t})=\operatorname{erfc}(\lambda\sqrt{\epsilon/2t})+\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{(-)^k}{(2k+1)2k!!}[(\lambda\sqrt{\epsilon/t}+a/\sqrt{2})^{2k+1}-(\lambda\sqrt{\epsilon/t})^{2k+1}] . \quad (C18)$$

We note that the second term of Eq. (C18) is $1/\sqrt{t}$ times a function of $\lambda/\sqrt{t}$ and consequently must be neglected in the spirit of our approximation. We now have a cutoff-independent result

$$G_\phi(\lambda,t)=2e^{\epsilon\lambda^2}\operatorname{erfc}(\lambda\sqrt{2\epsilon/t}) . \quad (C19)$$

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